# GAME THEORY

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## Aim

To teach the students the over view of game theory, classical examples of game theory and applications, history of game theory, classification of game theory, key elements in game theory, Mathematical concept used in game theory, Procedure for solving game theory problem and explain worked out example.

## Objective

Students should get clear idea on what is game theory and the underlying concepts, students should get though knowledge on the mathematical concept used in game theory, they should understand the solving technique and know what method can be applied to the given problem for getting solution.

## Outcome

At the end of the course student get the clear ides of the following , origin of game theory, Author who studied and developed game theory, Practical use of game theory in real life, types of game theory, Mathematic required for solving game theory, Technique of solving for different types of games.

## Preview of lecture

- ✓ Overview of game theory
- ✓ Classic examples of game theory
- ✓ History of game theory
- ✓ Classification of game theory
- ✓ Key elements in game theory
- ✓ Mathematical concept in Game Theory
- ✓ Procedure to solve 2x2 game without saddle point
- ✓ Algebraic Method for solving games without saddle point
- ✓ Home work and Assignment Problem
- ✓ References

## Overview of Game Theory

Game theory is an approach to modelling behaviour in situations where the outcome of your decisions *depends on the decisions of others*.

Game theory is the *study of strategic, interactive* decision making among rational individuals or organizations.

Game theory is a branch of *applied mathematics* that provides tools for analyzing situations in which parties ( *called players*) make decisions that are interdependent.

This interdependence causes each player to consider the other player's possible decisions( *or strategies*) in formulating strategy.

In addition, a player *need not be an individual*; it may be a nation, a corporation, or a team comprising many people with shared interests.

A solution to a game describes the optimal decisions of the players, who may have *similar*, *opposed, or mixed interests*, and the outcomes that may result from these decisions.

Game theory is applied for determining different strategies in the *business world*. It offers valuable tools for solving strategy problems.

## Classic examples of game theory

**(i) The Prisoner's Dilemma;** where two suspects are in police custody as accomplices for the same crime, but there is not enough evidence for a felony conviction. They are held and interrogated separately. If one prisoner testifies against the other while the other stays silent, the testifying prisoner goes free and the silent prisoner is convicted and serves ten years. If both prisoners stay silent, both are convicted on a minor charge and serve six years. If both prisoners testify against each other, each serves five years. How should the prisoners act? . The answer is that both prisoners should testify against the other, an outcome that is known as a *Nash equilibrium*. [i.e The players in each game depend on each other's rationality to make the optimal choices in every situation, thereby maximizing their own utility]

**(ii) Conflict of sales;** Let us assume that the client had a drug with dominant market share in a therapeutic area that is competitive, yet has few players. One of their competitors was behaving aggressively from a pricing perspective (e.g., continually matching and beating the client's price, even though the competitor gained little or no market share from such actions). The competitor was in the process of running a clinical trial to improve their product's label. The client wished to know how the competitor might behave depending on the client's next price move and the outcome of the clinical trial. The conclusion of the exercise was that engaging in a price war would not benefit any of the client or competitors. Therefore, keeping the price high and trading "market share for peace" would be the most profitable strategy.

**(iii) Advertising War: Coke vs. Pepsi :-** Without any advertising, each company earns $5b/year from Cola consumers. Each company can choose to spend $2b/year on advertising. Advertising does not increase total sales for Cola, but if one company advertises while the other does not, it captures $3b from the competitor

.

| | | Pepsi | |
|---|---|---|---|
| | | No Ad | Ad |
| | No Ad | $5b,$5b | $2b,$6b |
| Coke | Ad | $6b,$2b | $3b,$3b |

What will the Cola companies do? Is there a better feasible outcome

**(iii) In your everyday life**: Everything is a game, poker, chess, soccer, driving, dating, stock market advertising, setting prices, entering new markets, building a reputation bargaining, partnerships, job market search and screening designing contracts, auctions, insurance, environmental regulations international relations, trade agreements, electoral campaigns, Most modern economic research includes game theoretical elements.

**(iv)** Game theory has been used to analyze parlour games, but its applications are much broader.

## History of game theory

The individual closely associated with the creation of the theory of games is *John von Neumann*, one of the greatest mathematicians of this century. Although others proceeded him in formulating a theory of games - notably *Emile Borel* - it was *von Neumann* who published in 1928 the paper that laid the foundation for the theory of two-person zero-sum games.

The theory of Games was born in 1944 with the publication of *Theory of Games and Economic Behaviour* by Hungarian-born American mathematician *John von Neumann* and his Princeton University colleague *Oskar Morgenstern*, a German-born American economist. In their book, . They observed that economics is much like a game, wherein players anticipate each other's moves, and therefore requires a new kind of mathematics, which they called game theory. Their choice of title was a little unfortunate, since it quickly got shortened to "*Game Theory*,"

Nobel Laureate and a **Father of Game Theory**, Lloyd S. Shapley(92), who shared the 2012 Nobel Memorial Prize in Economic Science for work on **game theory** that has been used to study subjects as diverse as matching couples and allocating costs.

## Classification of game theory

It broadly classified into three main sub-categories of study

### (1) Classical game theory

It focuses on optimal play in situations where *one or more people* must make a decision and the impact of that decision and the decisions of those involved is known.

Decisions may be made by use of a *randomizing device like piping a coin* .

It focuses on questions like, What is my best decision in a given economic scenario, where a reward function provides a way for me to understand how my decision will impact my result.

**Examples:**

Poker, Strategic military decision making, Negotiations.

### (2) Combinatorial game theory

It focuses on optimal play in *two-player* games in which each player takes turns changing in *pre-defined ways*. In other word, combinatorial game theory does not consider games with chance (no randomness).

Generally two player strategic games played on boards. Moves change the structure of a game board.

**Examples:**

Chess, Checkers, Go.

### (3)Dynamic game theory:

It focuses on the analysis of games in which players must make decisions over time and in which those decisions will affect the outcome at the next moment in time.

It often relies on differential equations to model the behaviour of players over time.

It can help optimize the behaviour of unmanned vehicles or it can help you capture your baby sister who has escaped from her playpen.

In general games with time, Games with motion or a dynamic component.

**Examples:**

Optimal play in a dog fight, Chasing your brother across a room.

## Key elements in game theory

**Player:** who is interacting

**Strategies:** what are the options of each player? In what order do players act?

**Payoffs:** How do strategies translate into outcomes? What are players' preferences over possible outcomes?

**Information/Beliefs**: What do players know/believe about the situation and about one another? What actions do they observe before making decisions?

**Rationality:** How do players think?

# Mathematical concept in Game Theory

The following the prerequisite required for game theory

## Sum of gains and loss

If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the gains and losses equal zero, the corresponding game is said to be zero sum game.

## Types of games

Games can be classified according to certain significant features, the most obvious of which is the number of players. Thus, a game can be designated as being a *one-person, two-person, or n-person (with n greater than two) game*, games in each category having their own distinctive features.

## One-person games

One-person games are also known as *games against nature*. With no opponents, the player only needs to list available options and then choose the optimal outcome. When chance is involved the game might seem to be more complicated, but in principle the decision is still relatively simple.

For example, a person deciding whether to carry an umbrella. While this person may make the wrong decision, there does not exist a conscious opponent. That is, nature is presumed to be completely indifferent to the player's decision, and the person can base his decision on simple probabilities. One-person games hold little interest for game theorists.

## Two person zero sum game (with two players)

The game in which there are exactly *two player* and the interest of the players *completely opposed* are referred as two-person zero sum games. They are called zero-sum games because one player wins whatever the other player loses. In short it is denoted by TPZS game.

For example, All parlour game and sports, like Tic-tac-toe, chess, cribbage, backgammon, and tennis ect., are TPZS games

## Two person zero sum game (with more than two players)

TPZS games with *more than two people involved* are

(i) Team sports with only two sides, but with more than one player in each side

(ii) Many people involved in *surrogates for military conflict*, so it should come as no surprise that many military problems can also be analyzed as TPZS games.

## Games which are not TPZS

(i) Those parlour games in which the players cannot be clearly separated into two sides are not TPZS games

(ii) Those poker and Monopoly games when played by more than two people are not TPZS games.

(iii) Most real economic "games" are not TPZS, because there are too many players, and also the interests of the players are not completely opposed.

## Positive-sum game

In game theory, a term positive sum refers to situations in which the total of gains and losses *is greater than zero*.

A positive sum occurs when resources are somehow increased and an approach is formulated such that the desires and needs of all concerned are satisfied.

## Perfect games

Games of perfect information in which each player knows everything about the game at all times. It is called *perfect games*

For example, chess in which each player knows everything about the game at all times. In chess exactly one of three outcomes must occur if the players make optimal choices: (i) White wins (has a strategy that wins against any strategy of black), (ii) Black wins (iii) White and black draw.

## Imperfect games

Games of imperfect information in which each player do not knows everything about the game at all times. It is also called *imperfect games*

For example, Poker in which players do not know all of their opponents' cards.

### Finite games

Games in which each player has a finite number of options, the number of players is finite, and the game cannot go on indefinitely.

For example, chess, checkers, poker, and most parlour games are finite.

### Cooperative games

In game theory, a cooperative game (*or coalitional game*) is a game with competition between groups of players *("coalitions")* due to the possibility of *external enforcement* of cooperative behaviour (e.g. through contract law).

### Non cooperative games

Those are opposed to cooperative games in which there is either no possibility to forge alliances or all agreements need to be self-enforcing (e.g. through credible threats).

### Pay off

The outcome of the game resulting from a particular decision (or strategy) is called pay off . It is assumed that pay off is also known to the player in advance.

It is expressed in times of numerical values such as money, percent of market share or utility.

### Pay off matrix

The pay offs in terms of gains or losses, when players select their particular strategies, can be represented in the form of matrix is called pay off matrix.

Let $A_1$, $A_2$,....$A_m$ are possible strategies for player A

Let $B_1$,$B_2$,....$B_n$ are possible strategies for player B.

The total number of possible outcomes are m x n and it is assumed that each player knows not only his own list of possible course of action but also his opponent.

For our convenience, we assume that player A always a gainer whereas player B a looser.

Let $a_{ij}$=pay off which player A gain from player B, if player A choose strategy i and player B chooses strategy J.

Pay of matrix player A is represented in the form of table

| Player A's strategies | Player B's strategies | | | |
|---|---|---|---|---|
| | $B_1$ | $B_2$ | .... | $B_n$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | .... | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | .... | $a_{2n}$ |
| .... | | | .... | |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | .... | $a_{mn}$ |

## Remarks

For the zero sum games, the gain of one player is equal to the loss of other and vice versa. i.e one player pay off table would contain the same amounts in pay off table of other player with the sign changed. Therefore it is enough to construct pay of table for one player.

## Strategy

The strategy for a player is the list of all possible actions (moves or course of action) that he will take for every pay-off (outcome) that might arise. It is assumed that all course of possible actions are known in advance to the player.

## Types of Strategy

Usually player in game theory uses two types of strategy namely pure strategy and mixed strategy .

### (i) Pure strategy:

Particular course of action that are selected by player is called pure strategy (course of action). i.e each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other players strategy, and objective of the player is to maximize gain or minimize loss

### (ii) Mixed strategy:

Course of action that are to be selected on a particular occasion with some fixed probability are called mixed strategies. i.e there is a probabilistic situation and objective of the players is to maximize expected gain or minimize expected losses by making choice among pure strategy with fixed probabilities.

In mixed strategy, If there are 'n' number of pure strategies of the player, there exist a set $S=\{p_1,p_2,....p_n\}$ where $p_j$ is the probility with which the pure strategy, j would be selected and whose sum is unity.

i.e $p_1+p_2+.....+p_n=1$ and $p_j>=0$ for all j=1,2,...n.

### Remark:

**(i)** If a player randomly chooses a pure strategy, we say that the player is using a "mixed strategy." In a pure strategy a player chooses an action for sure, whereas in a mixed strategy, he chooses a probability distribution over the set of actions available to him.

**(ii)** If a particular $p_j=1$ and all others are zero, then the player is said to select pure strategy J.

### Optimal strategy

The particular strategy (or complete plan) by which a player optimizes his gains or losses without knowing the competitor's strategies is called optimal strategy.

### Value of the game

The expected outcome when players follow their optimal strategy is called the value of the game, It is denoted by V

### Basic assumptions of game

**(i)** Each player has available to him a finite number of possible strategies. The list may not be same for each player.

**(ii)** Player A attempts to maximize gains and player B minimize losses.

**(iii)** The decisions of both players are made individually prior to the play with no communication between them.

**(iv)** The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player 's decision.

**(v)** Both the players know not only possible pay offs to themselves but also other.

## Minmax-Maxmin principle

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. The objective of the study is to know how these players select their respective strategy so that they may optimize their pay off. Such a decision making criterion is referred to as the minmax -maxmin principle

## Remarks

Minmax-Maxmin principle given the best possible selection of strategy for both players in pure strategy problem.

## Saddle point

If the minmax value = maxmin value, then the game is said to have a saddle (equilibrium) point

## Remarks

**(i)** The corresponding strategy at saddle point are called optimum strategy.

**(ii)** The amount of pay off at an saddle point is known as the value of the game.

**(iii)** A game may have more than one saddle point.

**(iv)** There are game without saddle point.

**(v)** Its name derives from its being the minimum of a row that is also the maximum of a column in a payoff matrix—to be illustrated shortly—which corresponds to the shape of a saddle.

## Procedure to determine saddle point

### Step:-01

Select the minimum (lowest ) element in each row of the pay off matrix and write them under 'row minima' heading. Then select the largest element among these elements and enclose it in a rectangle. ▭

### Step:-02

Select the maximum (largest) element in each column of the pay of matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle. ◯

### Step:-03

Find out the elements which is same in the circle as well as rectangle and mark the position of such elements in the matrix. This element represents the value of the game and is called the saddle point.

## Games without saddle point

Suppose if there is no pure strategy solution for a game, then there is no saddle point exist. In these situations, to solve the game both the player must determine the *optimal mixtures of strategies* to find saddle point.

The optimum strategy mixture of each player may be determined by assigning each strategy *it probility of being chosen*. The optimal strategy so determined is called mixed strategy.

## Fair Game

If the value of the game is zero (i.e. there is no loss or gain for any player), the game is called fair game. The simplest type of game is one where the best strategies for both players are pure strategies. This is the case if and only if, the pay-off matrix contains a saddle point.

## Strictly determinable

A game is said to be strictly determinable if the maxmin and minmax values of the game are equal and both equal the value of the game.

## Example:-01 (Games with sadle point)

Find the optimal plan for both the player

| | | Player-B | | | |
|---|---|---|---|---|---|
| | | **I** | **II** | **III** | **IV** |
| Player-A | **I** | -2 | 0 | 0 | 5 |
| | **II** | 4 | 2 | 1 | 3 |
| | **III** | -4 | -3 | 0 | -2 |
| | **IV** | 5 | 3 | -4 | 2 |

## Solution:-

We use maxmin-minmax principle for solving the game.

| | | Player-B | | | | Row Minimum |
|---|---|---|---|---|---|---|
| | | **I** | **II** | **III** | **IV** | |
| Player-A | **I** | -2 | 0 | 0 | 5 | -2 |
| | **II** | 4 | 2 | $\boxed{\textcircled{1}}$ | 3 | $\boxed{1}$ |
| | **III** | -4 | -3 | 0 | -2 | -4 |
| | **IV** | 5 | 3 | -4 | 2 | -6 |
| Column Maximum | | 5 | 3 | ① | 5 | |

Select minimum from the column maximum values.

ie. Minimax = 1, (marked as circle )

Select maximum from the row minimum values

ie. Maximin = 1, ( marked as rectangle)

Player A will choose strategy II, which yields the maximum payoff of 1

Player B will choose strategy III.

The value of game is 1, which indicates that player A will gain 1 unit and player B will sacrifice 1 unit.

Since the maximin value = the minimax value =1, therefore the game has saddle point and the game is not fair game. (since value of the game is non zero)

Also maxmin=minimax=value of game, therefore the game is strictly determinable.

It is a pure strategy game and the saddle point is (A-II, B-III)

The optimal strategies for both players given by pure strategy , Player A must select strategy II and playerB must select strategy III.

**Example:-02**

For the game with payoff matrix

$$\begin{array}{cccc} & & \text{Player B} & \\ \text{Player A} & -1 & 2 & -2 \\ & 6 & 4 & -6 \end{array}$$

Determine the best strategies for players A and B and also the value of the game. Is this game (i) fair (ii) strictly determinable?

**Solution:-**

$$\begin{array}{ccccc} & & \text{Player B} & & \text{Row minimum} \\ \text{Player A} & -1 & 2 & \boxed{\enclose{circle}{-2}} & \boxed{-2} \\ & 6 & 4 & -6 & -6 \\ \text{Column maximum} & 6 & 4 & \enclose{circle}{-2} & \end{array}$$

Select minimum from the column maximum values.

ie. Minimax = 1, (marked as circle )

Select maximum from the row minimum values.

ie. Maximin = 1, ( marked as rectangle)

Player A will choose strategy I, which yields the maximum payoff of -2

Player B will choose strategy III.

The value of game is -2, which indicates that player A will gain -2 unit and player B will sacrifice -2 unit.

Since the maximin value = the minimax value =-2, therefore the game has saddle point and the game is not fair game. (since value of the game is non zero)

Also maxmin=minimax=value of game, therefore the game is strictly determinable.

It is a pure strategy game and the saddle point is (A-I, B-III)

The optimal strategies for both players given by pure strategy , Player A must select strategy II and playerB must select strategy III.

### Example:-03

Find the range of values of p and q which will render the entry (2,2) a saddle point for the game.

Player B

| Player A | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | 2 | 4 | 5 |
| $A_2$ | 10 | 7 | q |
| $A_3$ | 4 | p | 6 |

**Solution:-**

Let us ignore the values of p and q in pay off matrix and proceed to calculate maxmin and minmax values.

**Player B**

| **Player A** | **$B_1$** | **$B_2$** | **$B_3$** | Row minimum |
|---|---|---|---|---|
| **$A_1$** | 2 | 4 | 5 | 2 |
| **$A_2$** | 10 | $\boxed{7}$ | q | $\boxed{7}$ |
| **$A_3$** | 4 | p | ⑥ | 4 |
| Column Maximum | 10 | 7 | ⑥ | |

Here maxmin=7, minmax=6, i.e the value of the game may be between 6 and 7.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should solved using mixed strategy.

If the saddle point exist at (2,2), only if q>7 and p>=7.

### Example:-04

For what value of a, the game with following pay-offs matrix is strictly determinable?

Player B

| Player A | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $A_1$ | a | 6 | 2 |
| $A_2$ | -1 | a | -7 |
| $A_3$ | -2 | 4 | a |

**Solution:-**

Let us ignore the values of a in pay off matrix and proceed to calculate maxmin and minmax values.

|  | **Player B** |  |  |  |
|---|---|---|---|---|
| **Player A** | **$B_1$** | **$B_2$** | **$B_3$** | Row minimum |
| **$A_1$** | a | 6 | $\boxed{2}$ | $\boxed{2}$ |
| **$A_2$** | -1 | a | -7 | -7 |
| **$A_3$** | -2 | 4 | a | -2 |
| Column Maximum | -1 | 6 | 2 |  |

Here maxmin=2, minmax=-1, i.e the value of the game lies between -1 and 2.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

For strictly determinable game , we must have -1<=a<=2.

## Procedure to solve 2x2 game without saddle point

If pay-off matrix for player A is given by

|  |  |  |
|---|---|---|
|  | Player B |  |
| Player A | $a_{11}$ | $a_{12}$ |
|  | $a_{21}$ | $a_{22}$ |

then the following formulae are used to find the value of the game and optimal strategies.

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \& \quad p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \& \quad q_2 = 1 - q_1$$

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Hence the optimal mixed strategy for player A and B given by

$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ and $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

Here

$p_1$=probability of player A choose strategy $A_1$

$p_2$=probability of player A choose strategy $A_2$

$q_1$=probability of player A choose strategy $B_1$

$q_2$=probability of player A choose strategy $B_2$

**Example:-01**

Two player A and B match coins. If the coins match, then A wins two units of value, if the coin do not match, then B win 2 units of value. Determine the optimum strategies for the players and the value of the game

**Solution:-**

Let us construct the pay off matrix for player A

| | Player B | | |
|---|---|---|---|
| Player A | H | T | Row minimum |
| H | 2 | -2 | -2 |
| T | -2 | 2 | -2 |
| Column Maximums | 2 | 2 | |

Since maxmin=-2 and minmax=2. i.e the value of the game lies between -2 and 2. i.e maxmin is not equal to minmax, therefore there is no unique saddle point. Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the following formulae

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(2) - (-2)}{(2+2) - (-2-2)} = 4/8 = 1/2$$

$$p_2 = 1 - p_1 = 1/2$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{2 - (-2)}{2 + 2 - (-2 - 2)} = 4/8 = 1/2$$

$$q_2 = 1 - q_1 = 1/2$$

Therefore it is a fair game.

Hence the optimal mixed strategy for player A and B given by

$S_A = \begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$ and $S_B = \begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$

### Example:-02

Consider a modified form of " matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Rs. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

### Solution:-

Let us construct the pay off matrix for matching player

| | Non matching player | | |
|---|---|---|---|
| Matching Player | H | T | Row minimum |
| H | 8 | -3 | -3 |
| T | -3 | 1 | -3 |
| Column Maximums | 8 | 1 | |

Since maxmin=-3 and minmax=1. i.e the value of the game lies between -3 and 1.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the following formulae

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(1) - (-3)}{(8+1) - (-3-3)} = 4/15$$

$p_2 = 1 - p_1 = 11/15$

$$q_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(1) - (-3)}{(8+1) - (-3-3)} = 4/15$$

$q_2 = 1 - q_1 = 11/15$

Therefore it is a fair game.

Hence the optimal mixed strategy for player A and B given by

$S_A = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$ and $S_B = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$

# Algebraic Method for solving games without saddle point

Let $p_1,p_2,.....p_m$ be the probability that the player A choose his strategy $A_1$, $A_2$,...$A_m$ respectively. where $p_1+p_2+....+p_m=1$

Let $q_1,q_2,.....q_n$ be the probability that the player B choose his strategy $B_1$, $B_2$,...$B_n$ respectively. where $q_1+q_2+.....+q_n =1$

| Player A's strategies | Player B's strategies | | | | Probability |
|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | .... | $B_n$ | $p_1$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | .... | $a_{1n}$ | $p_2$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | .... | $a_{2n}$ | $p_3$ |
| .... | | | .... | | |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | .... | $a_{mn}$ | $p_m$ |
| Probability | $q_1$ | $q_2$ | | $q_n$ | |

Let V be the value of the game.

### To find $S_A$

The expected gain to player A when player B selects strategies $B_1$,$B_2$,...$B_n$ respectively

Since player A is gainer player and he expects at least V, therefore we have

$a_{11}p_1+a_{12}p_2+......+a_{m1}p_m>=V$

$a_{12}p_1+a_{22}p_2+....+a_{m1}p_m>=V$

......

$a_{1n}p_1+a_{22}p_2+....+a_{mn}p_m>=V$

To find the values of $p_i$'s , above inequalities are considered as equations and are then solved for given unknowns.

### To find $S_B$

The expected loss to player B when player A selects strategies $A_1$,$A_2$,...$a_m$ respectively

Since player B is loser player , therefore we have

$a_{11}q_1+a_{12}q_2+......+a_{1n}q_n<=V$

$a_{12q1}+a_{22}q_2+....+a_{m1}q_m>=V$

......

$a_{1n}q_1+a_{2n}q_2+....+a_{mn}q_m>=V$

To find the values of $q_i$'s , above inequalities are considered as equations and are then solved for given unknowns.

By substituting the values of $a_i$'s and $q_i$'s in any one of the above equation give the value of the game.

**Remarks:-**

This method becomes quite lengthy when number of strategies for both the players are more than two.

### Example:-01

Two player A and B match coins. If the coins match, then A wins two units of value, if the coin do not match, then B win 2 units of value. Determine the optimum strategies for the players and the value of the game

**Solution:-**

Let us construct the pay off matrix for player A

|  |  | Player B |  |  |
|---|---|---|---|---|
| Player A | H | T | Row minimum | Probability |
| H | ② | $\boxed{-2}$ | $\boxed{-2}$ | $p_1$ |
| T | $\boxed{-2}$ | ② | $\boxed{-2}$ | $p_2$ |
| Column Maximums | ② | ② |  |  |
| Probability | $q_1$ | $q_2$ |  |  |

Since maxmin=-2 and minmax=2. i.e the value of the game lies between -2 and 2. i.e maxmin is not equal to minmax, therefore there is no unique saddle point. Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the algebraic method

**To find $S_A$**

$2p_1-2p_2=V$ ----(1) and $-2p_1+2p_2=V$ ----(2)

Therefore we have $2p_1-2p_2 =-2p_1+2p_2$

$\Rightarrow 2p_1-2(1-p_1) =-2p_1+2(1-p_1)$ $[p_1+p_2=1]$

$\Rightarrow 2p_1-2+2p_1 =-2p_1+2-2p_1$

$\Rightarrow 2p_1-2+2p_1 =-2p_1+2-2p_1$

$\Rightarrow 8p_1=4$

$\Rightarrow p_1=1/2$

Thus $p_2=1-p_1=1-1/2=1/2$

### To find $S_B$

$2q_1-2q_2=V$ ----(3) and $-2q_1+2q_2=V$ ----(4)

Therefore we have $2q_1-2q_2 =-2q_1+2q_2$

$\Rightarrow 2q_1-2(1-q_1) =-2q_1+2(1-q_1)$ $[q_1+q_2=1]$

$\Rightarrow 2q_1-2+2q_1 =-2q_1+2-2q_1$

$\Rightarrow 2q_1-2+2q_1 =-2q_1+2-2q_1$

$\Rightarrow 8q_1=4$

$\Rightarrow q_1=1/2$

Thus $q_2=1-q_1=1-1/2=1/2$

Hence the optimal mixed strategy for player A and B given by

$S_A=\begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$ and $S_B=\begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$

Value of the game

$(1)\Rightarrow 2p_1-2p_2=V$

$\Rightarrow 2(1/2)-2(1/2)=V$

$\Rightarrow V=0$

**Remark:-**

To find the value of the game in the above problem we can also use equations (2), (3) and (4), we will get the same value for V. i.e $V=0$ (verify yourself)

## Example:-02

Solve the following 2x2 game

| | Player A | |
|---|---|---|
| Player B | 2 | 5 |
| | 7 | 3 |

## Solution:-

| | Player B | | | |
|---|---|---|---|---|
| Player A | H | T | Row minimum | Probability |
| H | $\boxed{2}$ | ⑤ | $\boxed{2}$ | $p_1$ |
| T | ⑦ | $\boxed{3}$ | $\boxed{3}$ | $p_2$ |
| Column Maximums | ⑦ | ⑤ | | |
| Probability | $q_1$ | $q_2$ | | |

Since maxmin=3 and minmax=5. i.e the value of the game lies between 3 and 5.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the algebraic method

### To find $S_B$

$2q_1+5q_2=V$ ---(1) and $7q_1+3q_2=V$----(2)

Therefore we have $2q_1+5q_2 =7q_1+3q_2$

$=>2q_1+5(1-q_1) =7q_1+3(1-q_1)$ $[q_1+q_2=1]$

$=>2q_1+5-5q_1 =7q_1+3-3q_1$

$=>5-3q_1 =4q_1+3$

$=>-7q_1=-2$

$=>q_1=2/7$

Thus $q_2=1-q_1=1-2/7=5/7$

### To find $S_A$

$2p_1+7p_2=V$ ---(3) and $5p_1+3p_2=V$---(4)

Therefore we have $2p_1+7p_2 =5p_1+3p_2$

=>$2p_1+7(1-p_1) =5p_1+3(1-p_1)$ [$p_1+p_2=1$]

=>$2p_1+7-7p_1 =5p_1+3-3p_1$

=>$7-5p_1 =2p_1+3$

=>$-7q_1=-4$

=>$p_1=4/7$

Thus $p_2=1-p_1=1-4/7=3/7$.

Hence the optimal mixed strategy for player A and B given by

$S_A=\begin{pmatrix} H & T \\ 4/7 & 3/7 \end{pmatrix}$ and $S_B=\begin{pmatrix} H & T \\ 2/7 & 5/7 \end{pmatrix}$

Value of the game

(4)=>$5p_1+3p_2=V$

=>$5(4/7)+3(3/7)=V$

=>$20/7+9/7=V$

=>$29/7=V$

## Home work and Assignment Problem

Solve the following game

1. Pay off matrix is

$$\begin{array}{llll} & \text{Player B} & & \\ \text{Player A} & 20 & 15 & 22 \\ & 35 & 45 & 40 \\ & 18 & 20 & 25 \end{array}$$

2. Pay off matrix is

$$\begin{array}{llll} & \text{Company B} & & \\ \text{Company A} & 40 & 45 & 50 \\ & 20 & 45 & 60 \\ & 25 & 30 & 30 \end{array}$$

3. Pay off matrix is

$$\begin{array}{lll} & \text{Player B} & \\ \text{Player A} & 2 & 5 \\ & 4 & 1 \end{array}$$

4. Pay off matrix is

$$\begin{array}{lll} & \text{Player B} & \\ \text{Player A} & 6 & 9 \\ & 8 & 4 \end{array}$$

5. Pay off matrix is

$$\begin{array}{lll} & \text{Player B} & \\ \text{Player A} & 1 & 1 \\ & 4 & -3 \end{array}$$

6. Pay off matrix is

$$\begin{array}{llll} & \text{Player B} & & \\ \text{Player A} & 1 & 3 & 1 \\ & 0 & -4 & -3 \\ & 1 & 5 & -1 \end{array}$$

7. Pay off matrix is

$$\begin{array}{llllll} & \text{Player B} & & & & \\ \text{Player A} & 9 & 3 & 1 & 8 & 0 \\ & 6 & 5 & 4 & 6 & 7 \\ & 2 & 4 & 3 & 3 & 8 \\ & 5 & 6 & 2 & 2 & 1 \end{array}$$

8. Pay off matrix is

$$\begin{array}{lll} & \text{Player B} & \\ \text{Player A} & 2 & -1 \\ & -1 & 0 \end{array}$$

9. Pay off matrix is

$$\begin{array}{lll} & \text{Player B} & \\ \text{Player A} & 5 & 1 \\ & 3 & 4 \end{array}$$

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### Some useful links

https://www.britannica.com/science/game-theory

https://www.google.com/search?client=firefox-b-d&q=game+theory+history

https://plato.stanford.edu/entries/game-theory/